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Exam I MTH 512 , Fall 2018

Ayman Badawi

QUESTION 1. Let $B = \{(1,-1,-1),(1,0,-1),(1,-1,0)\}$ be a basis for R^3 and $B' = \{(1,-1),(-1,2)\}$ be a basis for R^2. Let $T : R^3 \to R^2$ be a linear transformation over R such that $T(1,-1,-1) = (1,-1), T(1,0,-1) = (-1,1), T(1,-1,0) = (1,0)$.

(i) Find the matrix representation of T with respect to B and B', i.e. $M_{B,B'}$.

Using class notes $Q = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$, $W = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. **Let** M **be the standard matrix representation.**

We know $M_{B,B'} = W^{-1}MQ$. **Now, reading class notes, we know that** $T(1,-1,-1)$ **is the first column of** MQ, $T(1,0,-1)$ **is the second column of** MQ, **and** $T(1,-1,0)$ **is the third column of** MQ. **Hence staring at the question** $MQ = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. **Now** $W^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. **Hence** $M_{B,B'} = W^{-1}MQ = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. .

(ii) Find a general formula for $[(a,b,c)]_B$

by Class notes: $[(a,b,c)]_B = Q^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -a-b-c \\ a+b \\ a+c \end{bmatrix}$.

(iii) Find a general formula for $[(c,d)]_{B'}$ **By class notes:** $[(c,d)]_{B'} = W^{-1}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2c+d \\ c+d \end{bmatrix}$.

(iv) Find $[(2,1,-1)]_B$, $[T(2,1,-1)]_{B'}$, and $T(2,1,-1)$

Again, class notes: (a) From (II)$[(2,1,-1)]_B = Q^{-1}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$

(b) From class notes: $[T(2,1,-1)]_{B'} = M_{B,B'}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

(c) From (b) and using B', $T(2,1,-1) = -3(1,-1) + 1(-1,2) = (-4,5)$.

(v) Write Range T as span

The easiest is to use M **from (VII). Stare at** M. **Do minor row operations. Range =** $span\{(-1,2),(-2,2)\}$

(vi) write Z(T) (ker(T)) as span

Staring at M and read the homogenous system, we conclude $Z(T) = span\{(-1,0.5,1)\} = span\{(-2,1,2)\}$.

(vii) Find the standard matrix representation of T.

From (i), MQ **is given. Hence** $MQ = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. **Thus,** $M = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} Q^{-1} = \begin{bmatrix} -1 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$.

QUESTION 2. Let $T : P_4 \to P_3$ be a linear transformation over a field F such that $dim(Z(T)) = 2$ (i.e., dim(Ker(T)) = 2). Convince me that there is a polynomial $f(x) \in P_3$ such that $T(d(x)) \neq f(x)$ for every $d(x) \in P_4$.

Proof. We know $dim(Z(T)) + dim(Range(T)) = 4$. **Since** $dim(Z(T)) = 2$, **we conclude** $dim(Range(T)) = 2$. **Thus** T **is not onto. Hence** $Range(T)$ **is a proper subspace of** P^3. **Thus there is a polynomial** $f(x) \notin Range(T)$. **Hence for every** $d(x) \in P_4$ **we have** $T(d(x)) \neq f(x)$. .

QUESTION 3. Let $B = \{x^3+1, x^3+x^2+x+1\}$ be a basis of a subspace of P_4 over Q. Extend B to a basis of P_4 $(over\ Q)$. **use the techniques I gave you in class. Done. All of you got it right.** .

QUESTION 4. Let $T : P_4 \to P_4$ be a linear transformation over R such that $C_T(\alpha) = \alpha^4 - 4\alpha^2$. Find all possible polynomials in P_4, say $f(x)$, such that $T(f(x)) = -f(x)$

T **has only 0(repeated twice), 2, -2 as eigenvalues. Suppose there is a nonzero polynomial** $f(x)$ **in** P_4 **such that** $T(f(x)) = -f(x)$. **Then we conclude that** -1 **is an eigenvalue of** T. **Since** -1 **is not an eigenvalue of** T, **we conclude that** $T(f(x)) = -f(x)$ **if and only if** $f(x) = 0$.

QUESTION 5. Let $T : P_4 \to R^2$ be a linear transformation over R such that $T(f(x)) = (f(1), f(0))$.

1)Find Z(T) (Ker(T)) and write it as span.

Typical question: No one should miss, $Z(T) = span\{-x^3 + x^2, -x^3 + x\}$.

2) Write Range(T) as span

Since $dim(Z(T)) = 2$**, then** $dim(Range(T)) = 2$**. Hence** $Range(T) = R^2$ **(i.e., T is ONTO). Thus span of any two independent points in** $R^2 = Range(T) = R^2$**. For Example, we may say** $Range(T) = span\{(1, 0), (0, 1)\}$**.**

QUESTION 6. Let A be a 3×3 matrix over Q with eigenvalues 2, -1. Given $E_2 = span\{(1, 1, -1), (0, 1, -1)\}$ and $E_{-1} = span\{(-1, -1, 2)\}$.

1)Find a matrix D, 3×5, of maximum rank such that such that $AD - 2D = 0_{3\times5}$, where $0_{3\times5}$ is the zero-matrix 3×5.

Note, we need D **such that** $AD = 2D$**. This means:** $A\times$ **(first column of D) = 2 X (First column of D), and so on,** $A\times$ **(fifth column of D) = 2 X (fifth column of D). Hence by staring, columns of** D **must "live" in** E_2**. Since** $dim(E_2) = 2$**,** D **will have maximum two independent columns. Thus** $Rank(D) = 2$**. So you may take**

$$D = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 1 & 1 & 2 & 3 & 0 \\ -1 & -1 & -2 & -3 & 0 \end{bmatrix}.$$

2)Find $|A|$ and $Trace(A)$.

by Staring at E_2 **and** E_{-1}**,** $C_A(\alpha) = (\alpha - 2)^2(\alpha + 1)$**. Eigenvalues are: 2, 2, -1.**

Hence $|A| = 2X2X - 1 = -4$. $Trace(A) = 2 + 2 + -1 = 3$.

QUESTION 7. Let $T : P_2 \to P_2$ be a linear transformation over R such that $C_T(\alpha) = \alpha^2 - 4$.

1) Let $L : P_2 \to P_2$ be a linear transformation over R such that $L(f(x)) = T^2(f(x)) + 4T(f(x)) + f(x)$. Find $C_L(\alpha)$

Eigenvalues of T **are** $2, -2$**. Thus exist nonzero polynomials** $d(x), w(x) \in P_2$ **such that** $T(d(x)) = 2d(x)$ **and** $T(w(x)) = -2w(x)$**. Thus** $L(d(x)) = T^2(d(x)) + 4T(d(x)) + d(x) = 4d(x) + 8d(x) + d(x) = 13d(x)$**. Hence** 13 **is an eigenvalue of** L**.**

$L(w(x)) = T^2(w(x)) + 4T(w(x)) + w(x) = 4w(x) - 8w(x) + w(x) = -3w(x)$**. Thus -3 is an eigenvalue of** L**.**

Hence $C_L(\alpha) = (\alpha - 13)(\alpha + 3)$.

2)For every $f(x) \in P_2$, find $T^4(f(x))$

Clearly, $P_2 = span\{d(x), w(x)\}$ **(d(x), w(x) as in (1) above). Let** $f(x) \in P_2$**. Then** $f(x) = ad(x) + bw(x)$ **for some** $a, b \in R$**.**

Thus $T^4(f(x)) = T^4(ad(x) + bw(x)) = aT^4(d(x)) + bT^4(w(x)) = 16ad(x) + 16bw(x) = 16(ad(x) + bw(x)) = 16f(x)$.

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: `abadawi@aus.edu, www.ayman-badawi.com`